Fourier Analysis   Jan 25, 2022.

Review.

- A convergence Thm: If f is cts on the circle such that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$, then

$$S_N f(x) := \sum_{n=-N}^{N} \hat{f}(n) e^{inx} \rightrightarrows f(x) \quad \text{on the circle}$$

as $N \to \infty$.

- A sufficient condition for $\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$:

  f is twice continuously diff on the circle.

  (If so, $\exists\, M > 0$ such that

  $$|\hat{f}(n)| \leq \frac{M}{n^2} \quad \text{for all } n \neq 0.)$$

Remark: Even if f is cts on the circle, it is possible

$$S_N f(x) \not\to f(x)$$

at some point x. We will construct such an example in a later class.

§ 2.5 Convolutions.

Let us estimate $S_N f(x)$:

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)\, e^{inx}$$

$$= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, e^{-iny}\, dy \cdot e^{inx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \cdot \sum_{n=-N}^{N} e^{in(x-y)}\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, D_N(x-y)\, dy,$$

with $D_N(x) := \sum_{n=-N}^{N} e^{inx}$

(which is called the N-th Dirichlet kernel)

Write $f * D_N(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, D_N(x-y)\, dy$

We call the function $f * D_N$ the convolution of f and D_N.

Def (Convolution): Let f, g be two integrable functions on the circle. We define

$$f*g(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy.$$

Remark: For given x, $f(y)\, g(x-y)$ is an integrable function in y on the circle.

Prop 1: (1) $f*(g+h) = f*g + f*h$

(2) For $c \in \mathbb{C}$,

$$(cf)*g = c(f*g)$$

(3) $f*g = g*f$.

(4) $(f*g)*h = f*(g*h)$

(5) $f*g$ is cts on the circle.

(6) $\widehat{f*g}(n) = \hat{f}(n)\cdot\hat{g}(n), \quad \forall\, n \in \mathbb{Z}$.

Pf: We only prove ③, ⑤ and ⑥, and leave the others to you as an exercise.

③ $f * g = g * f$.

Recall that for a fixed x,

$$f*g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy$$

$$\xlongequal{\text{Letting } z=x-y} \frac{1}{2\pi}\int_{x+\pi}^{x-\pi} f(x-z)\, g(z)\, (-1)\, dz$$

$$= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-z)\, g(z)\, dz$$

(since $f(x-z)\,g(z)$ is 2π-periodic in z)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-z)\, g(z)\, dz$$

$$= g*f(x).$$

(6) $\widehat{f*g}(n) = \hat{f}(n)\, \hat{g}(n)$.

Notice that

$$\widehat{f*g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f*g(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,g(x-y)\,dy\right)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\cdot g(x-y)e^{-in(x-y)}\,dy\,dx$$

(Fubini Thm: Let $F(x,y)$ be a Riemann integrable function on $[a,b]\times[c,d]$. Then

$$\int_a^b\int_c^d F(x,y)\,dy\,dx = \int_c^d\int_a^b F(x,y)\,dx\,dy$$

$$= \iint_{[a,b]\times[c,d]} F(x,y)\,dx\,dy \quad)$$

(by Fubini)

$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\underbrace{\int_{-\pi}^{\pi} g(x-y)e^{-in(x-y)}\,dx}\, f(y)e^{-iny}\,dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{g}(n)\,f(y)e^{-iny}\,dy$$

$$= \hat{g}(n)\,\hat{f}(n).$$

(5) $f*g$ is cts on the circle.

We first prove the result in the case when g is cts on the circle.

Notice that g is uniformly cts on the circle.

Hence $\forall\ \varepsilon>0$, $\exists\ \delta>0$ such that

$$|g(y_1)-g(y_2)|<\varepsilon \quad \text{if} \quad |y_1-y_2|<\delta.$$

Now for x_1, x_2 on the circle with $|x_1-x_2|<\delta$,

$$\begin{aligned} f*g(x_1)-f*g(x_2) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,g(x_1-y)\,dy \\ &\quad - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,g(x_2-y)\,dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot\big(g(x_1-y)-g(x_2-y)\big)\,dy \end{aligned}$$

Hence

$$| f*g(x_1) - f*g(x_2)|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)| \cdot | g(x_1-y) - g(x_2-y)| \, dy$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)| \cdot \varepsilon \, dy$$

$$= \varepsilon \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)| \, dy$$

$$\leq \varepsilon \cdot B, \quad \text{where } B = \sup_{y\in[-\pi,\pi]} |f(y)| .$$

Hence $f*g$ is uniformly cts on the circle.

Next we consider the general case when g is merely integrable on the circle.

We need an auxiliary result.

<u>Lem 2</u>: Let g be integrable on $[-\pi, \pi]$.

Then for any $\varepsilon > 0$, $\exists$ a cts function h

on the circle such that

$$|h(x)| \leq \sup_{y \in [-\pi,\pi]} |g(y)| , \quad \forall \ x \in [-\pi, \pi]$$

and
$$\int_{-\pi}^{\pi} | g(x) - h(x) | \, dx < \varepsilon .$$

We postpone the proof of the above lemma a while.

Now we use it to prove the continuity of $f * g$.

By Lemma 2, we can take a sequence of cts functions (h_n) on the circle such that

$$\int_{-\pi}^{\pi} |g(x) - h_n(x)| \, dx \to 0 \quad \text{as } n \to \infty .$$

Now let us estimate

$$f*g(x) - f*h_n(x)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot h_n(x-y)\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot\big(g(x-y) - h_n(x-y)\big)\, dy.$$

Hence

$$|f*g(x) - f*h_n(x)|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)|\cdot|g(x-y) - h_n(x-y)|\, dy$$

$$\leq \frac{B}{2\pi}\int_{-\pi}^{\pi} |g(x-y) - h_n(x-y)|\, dy$$

(where $B := \sup_{y\in[-\pi,\pi]} |f(y)|$)

$$= \frac{B}{2\pi}\int_{-\pi}^{\pi} |g(y) - h_n(y)|\, dy$$

It implies that

$$f * h_n(x) \rightrightarrows f * g(x) \quad \text{on the circle}$$

as $n \to \infty$.

Since $f * h_n$ is cts on the circle, it follows that $f * g$ is cts on the circle.

$\blacksquare$

Now we turn to the proof of Lem 2.

Pf of Lem 2. Since g is Riemann integrable, $\forall \varepsilon > 0$, $\exists$ a partition

$$-\pi = x_0 < x_1 < \cdots < x_N = \pi$$

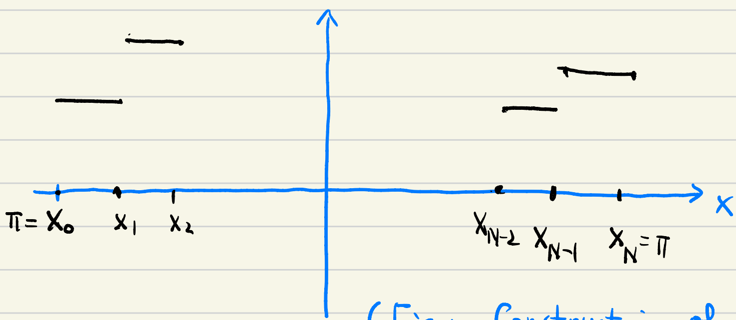
of $[-\pi, \pi]$ such that

$$\left| \int_{-\pi}^{\pi} g(x)\,dx - \overbrace{\sum_{i=1}^{N} \left(\inf_{y \in [x_{i-1}, x_i]} g(y) \right) (x_i - x_{i-1})}^{\text{Lower sum}} \right|$$

$$< \varepsilon$$

Define

$$g^*(x) = \inf_{y \in [x_{i-1}, x_i]} g(y) \quad \text{if} \quad x \in [x_{i-1}, x_i)$$



(Fig: Construction of g^*)

Notice that

- $g(x) \geq g^*(x) \quad \forall\, x \in [-\pi, \pi]$
- $|g^*(x)| \leq \sup_{y \in [-\pi, \pi]} |g(y)|$.

Recall that

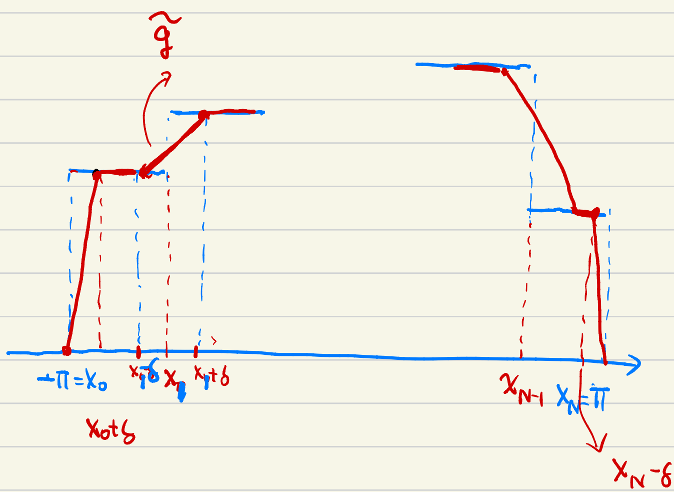
$$\left| \int_{-\pi}^{\pi} g(x)\,dx - \sum_{i=1}^{N} \left(\inf_{y \in [x_{i-1}, x_i]} g(y) \right) (x_i - x_{i-1}) \right|$$

$$= \left| \int_{-\pi}^{\pi} g(x)\,dx - \int_{-\pi}^{\pi} g^*(x)\,dx \right|$$

$$= \left| \int_{-\pi}^{\pi} g(x) - g^*(x)\,dx \right|$$

$$= \int_{-\pi}^{\pi} \left| g(x) - g^*(x) \right| dx \qquad \left(\text{since } g(x) \geq g^*(x) \right)$$

So $\int_{-\pi}^{\pi} \left| g(x) - g^*(x) \right| dx < \varepsilon$.



(Fig. Construction of $\tilde{g}$)

We choose a small number $\delta > 0$, and construct a function $\tilde{g}$ on $[-\pi, \pi]$ such that

① For each $1 \le i \le N-1$,

$\tilde{g}$ is linear on each δ-neighborhood of x_i

such that $\tilde{g}(x_i - \delta) = g^*(x_i - \delta)$, $\tilde{g}(x_i + \delta) = g^*(x_i + \delta)$

② $\tilde{g}$ is linear on $[-\pi, -\pi+\delta]$ such that

$$\tilde{g}(-\pi) = 0, \quad \tilde{g}(-\pi+\delta) = g^*(-\pi+\delta)$$

$\tilde{g}$ is linear on $[\pi-\delta, \pi]$ such that

$$\tilde{g}(\pi) = 0, \quad \tilde{g}(\pi-\delta) = g^*(\pi-\delta).$$

③ $\tilde{g}$ and g^* coincide except on these N-intervals.

Then $\tilde{g}$ is cts on the circle.

$$\int_{-\pi}^{\pi} |\tilde{g}(x) - g^*(x)|\,dx \leq 2N\delta \cdot 2B$$

where $B = \sup_{y \in [-\pi,\pi]} |g(y)|$

Hence

$$\int_{-\pi}^{\pi} |g(x) - \tilde{g}(x)|\,dx$$

$$\leq \int_{-\pi}^{\pi} |g(x) - g^*(x)|\,dx + \int_{-\pi}^{\pi} |g^*(x) - \tilde{g}(x)|\,dx$$

$$\leq \varepsilon + 2N\delta \cdot 2B$$

(choose δ small enough so that $2N\delta \cdot 2B < \varepsilon$)

$$\leq 2\varepsilon.$$

One can check $\sup|\tilde{g}| \leq \sup|g^*| \leq B$.

This completes the proof of Lem 2. ▨